A unified model for the generation and fission of internal tides in a rotating ocean

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ABSTRACT

A new two-layer model consisting of generalized Boussinesq equations is derived which contains forcing terms due to barotropic tidal flow over large-amplitude bottom topography. These equations can describe both the generation of nonlinear dispersive internal tides and their disintegration into solitary waves. Special attention is paid to the effects of Coriolis dispersion (which is due to the earth's rotation). Numerical solutions based on observed oceanic conditions show convincingly that the earth's rotation can be a decisive factor at mid-latitudes in that it tends to impede the disintegration of the internal tide. Oceanic observations in the Celtic Sea and in Massachusetts Bay are well reproduced by the model.

1. Introduction

In the past decades oceanic observations have yielded abundant evidence of the existence of oceanic internal solitary waves. They are usually associated with the soliton solution of the Korteweg-de Vries (KdV) equation (Osborne and Burch, 1980; Apel et al., 1985; Pingree and Mardell, 1985; Ostrovsky and Stepanyants, 1989). The two essential ingredients of this equation are nonlinearity and nonhydrostatic dispersion; thanks to a balance between these effects the soliton retains its shape while propagating. The KdV-soliton solution, besides having the characteristic hyperbolic secans shape, obeys specific relationships between phase speed, amplitude and length scale, which are usually satisfactorily met by the observed oceanic internal solitary waves (Ostrovsky and Stepanyants, 1989). All these aspects, however, relate only to the propagation of the solitary waves.

How they are generated is a different question. The observations indicate that they originate from a barotropic tidal flow over a topographic feature, like a large bank or continental slope. This generation mechanism, however, is indirect: a barotropic tidal flow over topography generates internal tides, which propagate away from the area of generation; during the course of their propagation they may disintegrate to form sequences of internal solitary waves (Sandstrom and Elliott, 1984; Holloway, 1987; Smyth and Holloway, 1988; Gerkema and Zimmerman, 1995). The latter paper
focussed on the conditions under which such a disintegration occurs. It turned out
that, besides the obvious role of nonlinearity and nonhydrostatic dispersion, a second
dispersive mechanism (referred to as Coriolis dispersion), which is due to the earth's
rotation, can play a decisive role in this process in that it tends to impede the
disintegration of the internal tide into solitary waves.

This was shown by solving (numerically) a set of forced rotation-modified Bouss-
inesq equations, which can describe both the generation of the internal tides and
their disintegration into solitary waves. However, the model could not be applied to
oceanic observations, because it was assumed that the topography, though of finite
amplitude, was small compared to the total fluid depth; in nature internal tides with a
pronounced nonlinear behavior are generated at topographies whose amplitudes are
of the same order of magnitude as the total fluid depth, e.g., a large bank or a
continental slope. So the question remains as to what happens in oceanic conditions:
whether the earth's rotation has a significant effect on the generation of oceanic
internal solitary waves.

It is the goal of this paper to explore this question by deriving a new set of forced
rotation-modified Boussinesq equations, in which the topography is allowed to be
arbitrarily large. The model consists of the following elements: (1) Density stratifica-
tion (two-layer model), (2) Nonlinearity, (3) Nonhydrostatic dispersion, (4) Coriolis
dispersion, (5) A large-amplitude bottom topography, (6) A barotropic tidal flow.

The earth's rotation induces a velocity component in the transverse direction; for
simplicity we assume that the problem is uniform in this direction. The last two
aspects (5 and 6) combine to form the generation mechanism of the internal tide.

Using these equations, we simulate the observations in the Celtic Sea by Pingree
and Mardell (1985), and in Massachusetts Bay by Halpern (1971), see Section 3.
Notice that both sites are located at mid-latitudes; the effect of the earth's rotation is
thus likely to be important.

A model encompassing all these elements is nonexistent in the literature. How-
ever, various interesting subcases have been studied before. If one omits the
generation mechanism, the equations reduce to rotation-modified Boussinesq or
KdV equations (Ostrovsky, 1978; Leonov, 1981; Grimshaw, 1985; Tomasson and
Melville, 1992). Leonov showed that form-preserving solitary waves cannot exist in
the presence of rotation; we discuss this point in Section 4.

A further reduction can be made by applying the hydrostatic approximation
(long-wave limit); one can then study whether nonlinearity and Coriolis dispersion
are able to balance. It was shown (Ostrovsky, 1978) that this is the case if the Coriolis
parameter is sufficiently large; in this case a periodic (but no solitary) form-
preserving solution exists. This regime is important in the present study, because it
corresponds to the case in which the earth's rotation prevents the internal tide from
disintegrating.

It should be emphasized that the oceanic internal solitary waves are generated
(though indirectly) by an oscillating flow over topography, which makes the problem essentially different from studies of the direct generation of solitary waves by a stationary flow over topography (see the review paper by Helfrich and Melville, 1990).

The importance of the time-dependence in the forcing was also emphasized in recent papers by Sandstrom and Quon (1993, 1994), who treated the generation and further evolution of the internal tide as separate problems. The generation was regarded as a hydrostatic process; the resulting profile was used as a starting point of another, nonhydrostatic model in which disintegration into solitary waves can occur. As explained above, our model consists of one single set of equations which captures both processes at once. A further difference is our inclusion of the earth's rotation.

In a recent study Lamb (1994) solved the fully nonlinear, nonhydrostatic equations\(^2\) on the f-plane numerically to make extensive simulations of the generation of internal tides and solitary waves near Georges Bank. Lamb's approach is preferable in cases in which a two-layer model cannot be used, or which involve strong nonlinearities. Our model, however, has the advantage of being built around the well-understood framework of the KdV and nonlinear Klein-Gordon equations, which provide us with a tool for further interpretation of the numerical results.

Finally, in some earlier studies on the generation of nonlinear internal tides, the appearance of solitary waves was excluded because the models were based on the hydrostatic approximation (Maze, 1987; Willmott and Edwards, 1987; and Heathershaw et al., 1987).

This paper is organized as follows. In Section 2 we introduce the nondimensional parameters, the scaling and the new set forced rotation-modified Boussinesq equations (the derivation of which is given in an appendix). In Section 3 we present the numerical solutions of the equations for the cases described in Halpern (1971) and Pingree and Mardell (1985). We end with a summary and discussion in Section 4.

2. The internal-tide generation model

a. Definitions. One of the central parameters of the problem is that of nonlinearity, an appropriate definition of which is:

\[
\epsilon = \frac{\text{tidal excursion amplitude}}{\text{topographic length scale}} \times \frac{\text{topographic height}}{\text{fluid depth}}. \tag{1}
\]

Since we wish to consider the generation of internal tides by tidal flow over a large-amplitude topography, we must allow the second ratio (denoted by \(\epsilon_0\)) on the right-hand side to be of order one. Furthermore, we assume that the internal waves

2. Note that Lamb uses the term “Boussinesq equations” to describe the equations of motion in which the Boussinesq approximation has been made. We use the term in the entirely different sense of the KdV relatives which allow wave propagation in both directions, see Whitham (1974, Section 13.11).
are weakly nonlinear (i.e., \( \epsilon \) small), so that we can derive generalized Boussinesq equations by using a perturbation approach. We thus have

\[
\epsilon_b = O(1) \quad \epsilon \ll 1.
\]  

Let \( H \) be the maximum depth of the system; then \((1 - \epsilon_b)H\) denotes the minimum depth.\(^3\) Furthermore, let \( U_0 \) be the amplitude of the barotropic tidal current in the deepest region (where the depth is \( H \)); then its maximum is \( U_0/(1 - \epsilon_b) \) in the shallowest region. We define

\[
c = (\Delta \rho g(1 - \epsilon_b)H)^{1/2}
\]

which is a measure of the phase speed of the internal tides. Here \( \Delta \rho \) is the relative difference in density between the layers: \( \Delta \rho = (\rho_2 - \rho_1)/\rho \), where \( \rho_1 \) (\( \rho_2 \)) denotes the density of the upper (lower) layer, and \( \rho = \frac{1}{2}(\rho_1 + \rho_2) \) the mean density.

The essential nondimensional parameters, which govern the generation and propagation of the internal tide, are now defined as follows:

\[
F = \frac{U_0/(1 - \epsilon_b)}{c} \quad \Lambda = \frac{c}{\sigma L_b} \quad \epsilon = F \Lambda \epsilon_b \quad \delta = \left(\frac{\sigma H}{c}\right)^2 \quad \mu = \left(\frac{f}{\sigma}\right)^2
\]

where \( L_b \) is the length scale of the topography, \( \sigma \) the tidal frequency, and \( f \) the Coriolis parameter; \( \delta \) is a parameter of nonhydrostatic dispersion, \( \mu \) of Coriolis dispersion.

Now we have to make some assumptions about the order of magnitude of the parameters. Since \( \epsilon \) must be a small parameter, either \( F \) or \( \Lambda \), or both of them, must be chosen small (notice that \( \epsilon = O(F \Lambda) \), \( F \Lambda \) denoting the first ratio on the right-hand side of (1)). The most natural choice would probably be: \( \Lambda = O(1) \), \( F = O(\epsilon) \) (equal length scales for internal tide and bottom topography; weak barotropic tidal flow). However, the derivation would then turn out to be formidable. The reason for this is as follows. At the solitary-wave length scale (at which nonlinear and nonhydrostatic terms become important) variations in depth are usually very small. We wish to exploit this fact by regarding the coefficients of the nonlinear and nonhydrostatic terms (which turn out to depend on the bottom topography) as constant while performing manipulations like differentiating. This would greatly simplify the derivation. But how can we express this idea properly in terms of small parameters? One way is to introduce a separate (short) length scale for the solitary waves, and to perform a multiple-scale analysis. A slightly less proper, but much simpler way is to choose \( \Lambda \) ‘sufficiently small’, e.g. \( \Lambda = O(\epsilon^{1/2}) \). As is shown in the Appendix, the coefficients can then be handled in a simple way (i.e., regarded as constant with respect to differentiation) up to the order under consideration. So this choice indeed helps us to achieve our goal, and it turns out that there are no harmful side-effects.

\(^3\) In case of a continental slope, \( H \) is the depth in the ocean, and \((1 - \epsilon_b)H\) the depth on the shelf.
Furthermore, the parameters of nonhydrostatic and Coriolis dispersion must be related to the parameter of nonlinearity. As regards the former, the usual choice is that $\delta$ and $\epsilon$ be of the same order (KdV theory). A natural choice for $\mu$ would be $\mu = O(1)$, but this would necessitate taking into consideration terms of coupled dispersion (i.e., terms with $\delta \mu$). We wish to exclude such terms because the two dispersive mechanisms work on vastly different length scales. This can be achieved by choosing for instance $\mu = O(\epsilon^{1/2})$, so that $\mu \delta = O(\epsilon^{3/2})$, which is beyond the order under consideration. (Again, the same result would follow if we introduce a solitary-wave length scale and perform a multiple-scale analysis which, however, would make the derivation awkward and more complicated.)

So, in addition to (2), we make the following assumptions:

$$F, \Lambda, \mu = O(\epsilon^{1/2}) \quad \delta = O(\epsilon).$$

Furthermore, we apply the Boussinesq approximation (i.e., we replace $\rho_i$ with $\rho$ so far as the inertial mass is concerned).

**b. The equations of motion; scaling.** In the two-layer system, with the interface at rest being located at $z = 0$, and the rigid-lid at $z = \alpha H$, let the bottom be described by

$$z = -(1 - \alpha)H + \epsilon_b h(X) \quad X = x/L_B.$$  

where we assume that $h, h_x = O(1)$. Recall that the system is assumed to be uniform in the transverse direction. We prescribe the velocity components of the barotropic tide as follows:

$$U = \frac{U_0 \sin \sigma t}{1 - \epsilon_b h},$$

$$V = \frac{(f/\sigma)U_0 \cos \sigma t}{1 - \epsilon_b h},$$

$$W = (\alpha H - z)\epsilon_b h_x \frac{U_0 \sin \sigma t}{(1 - \epsilon_b h)^2}.$$

They obey the following linear, hydrostatic equations:

$$U_t - fV = -P_x/\rho \quad V_t + fU = 0 \quad U_x + W_z = 0$$

for an appropriate pressure term $P$. In addition, the following boundary conditions are satisfied:

$$W|_{z=-\alpha H} = 0 \quad W|_{z=-(1-\alpha)H+c_yfH} = \epsilon_b HU h_x.$$  

We define $\eta$ as the displacement of the interface from its rest level. Furthermore, $u_t$, $v_t$, and $w_t$ are the baroclinic velocity components in the $x$, $y$ (i.e. transverse), and $z$
direction (with $i = 1, 2$ for the upper and lower layer, respectively). We write the pressure as:

$$\dot{p}_i = P(t, x) + \rho g \alpha \bar{H} - \rho g z + p_i(t, x, z).$$  \hspace{1cm} (10)

We define $\bar{u}$ and $\bar{v}$ as the vertical shear at the interface of the horizontal velocity components in the propagation and the transverse direction, respectively:

$$\bar{u} = \lim_{z \downarrow \eta} u_2 - \lim_{z \downarrow \eta} u_1$$  \hspace{1cm} (11)

$$\bar{v} = \lim_{z \downarrow \eta} v_2 - \lim_{z \downarrow \eta} v_1$$  \hspace{1cm} (12)

On the isotropic $f$-plane, the equations of motion for the internal-wave field now read (in the Boussinesq approximation):

$$\rho(u_{ix} + (U + u_i)u_{ix} + u_iU_x + (W + w_i)u_{ix} - f u_i) = -\rho u_{ix}$$  \hspace{1cm} (13)

$$v_{ix} + (U + u_i)v_{ix} + u_iV_x + (W + w_i)v_{ix} + f u_i = 0$$  \hspace{1cm} (14)

$$\rho(w_{ix} + (U + u_i)w_{ix} + u_iW_x + (W + w_i)w_{ix} + w_iW_z) = -\rho w_{ix}$$  \hspace{1cm} (15)

$$u_{ix} + w_{iz} = 0$$  \hspace{1cm} (16)

and the boundary conditions:

$$w_1 = 0 \quad \text{at} \quad z = \alpha H$$  \hspace{1cm} (17)

$$w_2 = \epsilon_b H u_2 h_x \quad \text{at} \quad z = -(1 - \alpha)H + \epsilon_b H h$$  \hspace{1cm} (18)

$$W + w_i = \eta_t + (U + u_i)\eta_x \quad \text{at} \quad z = \eta$$  \hspace{1cm} (19)

$$p_2 - p_1 = (p_2 - p_1)g \eta \quad \text{at} \quad z = \eta.$$  \hspace{1cm} (20)

As the barotropic flow is forced to move over the topography, a vertical component ($W$) is induced, see the second expression in (9), which gradually becomes weaker at higher positions, and finally vanishes at the rigid lid. Of particular importance is the value of $W$ at the interface, denoted by $W^*$, which is a measure of the forcing. In linearized form this quantity is given by

$$W|_{z=0} = (1 - \epsilon_b)\epsilon c \sqrt{\delta} \frac{\alpha h_x}{(1 - \epsilon_b h)^2} \sin \alpha t.$$  

For the definition of the parameters, see (3) and (4). This quantity is (apart from the factor $\alpha h_x/(1 - \epsilon_b h)^2$, which in practical cases is of order one) of the order of $(1 - \epsilon_b)\epsilon c \sqrt{\delta}$. We use this scale, which is the pertinent scale for the generation of internal tides, to put $W$ (and $w_i$, too) into a nondimensional form. As temporal and spatial scales we use

$$[t] = 1/\sigma \quad [x] = c/\sigma \quad [z] = H.$$
Gerkema: Forced rotation-modified Boussinesq equations

It is now easy to find the scales of the other dynamical variables. According to (19), \( W \) and \( \eta_r \) must be of the same order, hence \( [\eta] = (1 - \epsilon_b)\epsilon H \). At the shallowest region (on the shelf, say) the depth is \((1 - \epsilon_b)H\), and if we scale according to this depth we find \( [w_{iz}] = (1 - \epsilon_b)\epsilon c\sqrt{\delta}/(1 - \epsilon_b)H = \epsilon \delta \), so \([u_i] = \epsilon c \) (continuity equation). From the momentum equations follow \([p_i] = \epsilon \rho c^2 \) and \([v_i] = \epsilon c \sqrt{\mu} \). Finally, the external field (barotropic tide) is scaled as

\[
[U] = Fc \quad [V] = Fc \sqrt{\mu} \quad [W] = (1 - \epsilon_b)\epsilon c \sqrt{\delta}.
\]

The last expression was in fact our starting point of the scaling procedure.

Using these scales, the equations of motion, (13) to (16), become in nondimensional form:

\[
u_{i,t} + (FU + \epsilon u_i)u_{i,x} + F\Lambda u_i U_X + \epsilon(1 - \epsilon_b)(W + w_i)u_{i,z} - \mu v_i = -p_{i,x} \quad (21)
\]

\[
u_{i,t} + (FU + \epsilon u_i)v_{i,x} + F\Lambda u_i V_X + \epsilon(1 - \epsilon_b)(W + w_i)v_{i,z} + u_i = 0 \quad (22)
\]

\[
(1 - \epsilon_b)\delta(w_{i,z} + (FU + \epsilon u_i)w_{i,x} + \epsilon\Lambda u_i W_X + \epsilon(1 - \epsilon_b)(W + w_i)w_{i,z} + \epsilon(1 - \epsilon_b)w_i W_z) = -p_{i,z} \quad (23)
\]

\[
u_{i,x} + (1 - \epsilon_b)w_{i,z} = 0 \quad (24)
\]

and the boundary conditions, (17) to (20):

\[
\text{at } z = \alpha \quad (25)
\]

\[
\text{at } z = \alpha - 1 + \epsilon_h h \quad (26)
\]

\[
W + w_i = \eta_r + (FU + \epsilon u_i)\eta_x \quad \text{at } z = (1 - \epsilon_b)\epsilon \eta \quad (27)
\]

\[
p_2 - p_1 = \eta \quad \text{at } z = (1 - \epsilon_b)\epsilon \eta \quad (28)
\]

where \( h = h(X) \), with \( X = \Lambda x; U, V \) and \( W \) are now given by:

\[
U = \frac{1 - \epsilon_b}{1 - \epsilon_b h} \sin t \quad (29)
\]

\[
V = \frac{1 - \epsilon_b}{1 - \epsilon_b h} \cos t \quad (30)
\]

\[
W = \frac{(\alpha - z) \sin t}{(1 - \epsilon_b h)^2} h_X. \quad (31)
\]

The interfacial value of the vertical component is now denoted as \( W* = W|_{z=(1-\epsilon_b)\epsilon \eta} \).

c. The forced rotation-modified Boussinesq equations. In the Appendix we show how (21) to (28) can be reduced by a formal expansion in \( \epsilon^{1/2} \). The first three orders yield a set of three equations from which \( \bar{u}, \bar{v} \) and \( \eta \) are to be solved for given topography.
\( h(X) \) and tidal flow \( U, V, W \):

\[
\begin{align*}
\ddot{u}_t + FU \ddot{u}_x + F\ddot{u}_x + \epsilon \frac{2\alpha - 1 + \epsilon_b h}{1 - \epsilon_b h} \dddot{u}_x - \mu \dddot{v} + \eta_x &= 0 \quad (32) \\
\ddot{v}_t + FU \ddot{v}_x + F\ddot{v}_x + \epsilon \frac{2\alpha - 1 + \epsilon_b h}{1 - \epsilon_b h} \dddot{v}_x - \mu \dddot{u} + \eta_x &= 0 \quad (33) \\
\eta_t + FU \eta_x - W^* + \epsilon \frac{2\alpha - 1 + \epsilon_b h}{1 - \epsilon_b h} (\dddot{u}_x) - \frac{\alpha(1 - \alpha - \epsilon_b h)}{(1 - \epsilon_b)(1 - \epsilon_b h)} \dddot{u}_x \\
&- \frac{\alpha^2 \epsilon_b \dddot{u}h_x}{(1 - \epsilon_b)(1 - \epsilon_b h)^2} - \frac{1}{3} \delta \alpha(1 - \alpha - \epsilon_b h)(\eta_t + FU \eta_x)_{xx} = 0 \quad (34)
\end{align*}
\]

which we refer to as the forced rotation-modified Boussinesq equations.

The origin of the terms is readily identified. The variation in depth gives rise to a modification of various coefficients (recognizable by \( \epsilon_b h \)), to an extra term (the penultimate term in (34), which describes the effects of varying topography on the propagation of internal waves), and, in combination with barotropic tidal flow, to the terms \( F\ddot{u}_x, F\ddot{v}_x \), and most importantly, to the forcing term \( W^* \). Also, there are extra advection terms due to barotropic tidal flow \( (FU(-\lambda)) \). Coriolis effects give rise to the penultimate term in (32), and to Eq. (33).

The standard Boussinesq equations for internal waves, with the usual nonlinear \( (\epsilon) \) and nonhydrostatic \( (\delta) \) terms, can be obtained from (32) and (34) by omitting variations in depth, barotropic tidal currents, and Coriolis effects.

3. Comparison with oceanic observations

In this section we present numerical solutions of (32), (33) and (34), in which the parameters of (3) and (4), and the bottom profile \( h(X) \) (with \( X = \Lambda x \)), are chosen in accordance with oceanic observations, namely in the Celtic Sea and in Massachusetts Bay.

We introduce a grid in time and space

\[ t_n = n\Delta t \quad x_j = j\Delta x \]

for integer values of \( n \) and \( j \); hereafter we write \( \eta^n \) for \( \eta(t_n, x_j) \), and similarly for \( \dot{u} \) and \( \ddot{v} \). For the various derivatives we use centered difference approximations, which are of second order:

\[
\begin{align*}
\frac{\partial \eta}{\partial t} (t_n, x_j) &\triangleq \frac{\eta^{n+1} - \eta^{n-1}}{2\Delta t} \\
\frac{\partial \eta}{\partial x} (t_n, x_j) &\triangleq \frac{\eta^{n+1} - \eta^{n-1}}{2\Delta x} \\
\frac{\partial^2 \eta}{\partial x^2} (t_n, x_j) &\triangleq \frac{\eta^{n+1} - 2\eta^n + \eta^{n-1}}{(\Delta x)^2}
\end{align*}
\]
Eqs. (32) and (33) can now be solved in a straightforward way, because $\mathbf{u}_{n+1}^{r}$ and $\mathbf{u}_{n+1}^{h}$ can be expressed in terms of known quantities at time level $n$. Eq. (34) leads to an implicit equation which can be solved by the method of forward/backward substitution. In all experiments we start with a system at rest; at $t = 0$ the barotropic tidal flow is turned on, and the generation of internal tides begins.

a. The Celtic Sea. In this section we focus on observations made by Pingree and Mardell (1985). They presented results obtained with a thermistor chain (Fig. 1), which was installed on the continental shelf of the Celtic Sea, about 25 km from the shelf break ($47^\circ 41.8'N$, $6^\circ 18.2'W$). These measurements are notable because they included both spring and neap tides. During spring tides, the amplitude of the internal tide (taken from trough to crest) reaches values between 50 and 60 m (22-24 July). Also, the internal tide seems to start disintegrating into solitary waves. By contrast, during neap tides (16-18 July) the trough-crest amplitude lies between 13 and 17 m and, apart from small irregularities, the signal looks almost sinusoidal.

To simulate the observations, we have to specify the stratification (that is, to distill a two-layer system from the observed density distribution), the bottom topography, and the barotropic tidal flow.

The paper contains data on the stratification. Using these data we find that the first internal mode has a phase speed of about 0.50 m/s, and that maximum isopycnal displacements occur at a depth of 50 m. We require that the internal mode in the two-layer model behaves similarly. Hence, the interface is to be located at 50 m depth, and the relative difference in density between the layers, $\Delta \rho$, should be 0.0007.
Figure 2. The topography, with indications of the shelf break (A) and the thermistor chain position (B) (25 km from the break). The dotted line shows $W|_{x=0}$ at maximum flood, see Eq. (8).

The latter value follows from the well-known formula for the phase speed of the internal mode in a two-layer system:

$$c_0^2 = \Delta \rho g \frac{h_1 h_2}{h_1 + h_2}.$$  \quad (35)

Here $h_1$ ($h_2$) denotes the upper (lower) layer thickness. We had already found that $h_1 = 50$ m, and since the local depth is 170 m, $h_2 = 120$ m. Now $\Delta \rho$ follows by setting $c_0 = 0.5$ m/s.

We take $H = 4000$ m as the depth of the ocean, so $\alpha = 50/4000 = 0.0125$. The depth on the shelf was about 170 m, so $\epsilon_s = (4000 - 170)/4000 = 0.9575$. Hence we find from Eq. (3): $c = 1.1$ m/s. Furthermore, the frequency of the tide (M2) is $\sigma = 2\pi/(12.42 \cdot 3600) = 1.41 \cdot 10^{-4}$ rad/s. One unit of the scaled, dimensionless $x$ thus corresponds to 7.8 km ($= c/\sigma$). We describe the continental slope by

$$h(X) = \begin{cases} 0 & X \leq -\pi/4 \\ \frac{1}{2}(1 + \sin 2X) & -\pi/4 \leq X \leq \pi/4 \\ 1 & X \geq \pi/4 \end{cases}$$

where $X = \Lambda x$. With this choice we have $h, h_x = O(1)$. In terms of $X$, the length of the slope is $\pi/2$. In the area in question, the length of the slope is about 50 km, which corresponds to 6.4 units in terms of $x$. Hence, by division, $\Lambda = X/x = (\pi/2)/6.4 \approx 0.25$. The break (defined as the location where the depth is 200 m) is located at $x = 2.8$, the thermistor chain (25 km from the break) at $x = 6.0$, see Figure 2. Also shown in this figure is the maximum vertical velocity of the barotropic tide (dotted line), see (8), taken at the rest level of the interface. In the region where this quantity is largest, the internal tides are most effectively generated; obviously this is near the break.
According to Pingree and Mardell, the maximum barotropic on-shelf current speed (i.e. $U_0/(1 - \epsilon_b)$) is 0.7 m/s, at spring tides. For the same region, a paper by Heathershaw (1985) contains data based on an average over the spring-neap cycle, yielding values of about 0.5 m/s. This suggests that the maximum barotropic on-shelf current speed at neap tides is about 0.3 m/s. We use these values to calculate $F$ and $\epsilon$, as defined by (4):

\[
\begin{array}{c|c|c}
 & \text{spring tides} & \text{neap tides} \\
\hline
U_0/(1 - \epsilon_b) & 0.7 \text{ m/s} & 0.3 \text{ m/s} \\
F & 0.64 & 0.27 \\
\epsilon & 0.15 & 0.065
\end{array}
\]

Finally, we find from the above-mentioned values: $\delta = 0.26$ (the parameter of nonhydrostatic dispersion), and $f = 1.08 \cdot 10^{-4} \text{ rad/s}$, so that $\mu = 0.59$ (the parameter of Coriolis dispersion). In all numerical experiments presented in this section, we use steps of the following magnitude: $\Delta x = 0.002$ and $\Delta t = 0.0005$, corresponding to dimensional values of about 16 m and 3.5 s, respectively.

In the first experiment we consider the generation of internal tides during spring tides ($F = 0.64$, $\epsilon = 0.15$). As we start with a system at rest, it takes some time before the signal has become periodic in time at a given position. In particular, due to the earth’s rotation the near-inertial waves have an almost vanishing group velocity, hence transients will continue to be important during a considerable time. At the position in question, i.e. $x = 6.0$, where the ‘thermistor chain’ is installed, the signal has become fairly periodic after about 3 tidal periods (notice that in terms of $t$, which is dimensionless, a tidal period takes $2\pi$ units). Figure 3 shows the evolution of the interface in time, at $x = 6.0$, during two tidal cycles. The overall shape is fairly similar to the real observation shown in Figure 1 (22–24 July). Moreover, if we calculate the
distance from trough to crest, we find about 60 m, which is in excellent agreement with the observed values, which lie between 50 and 60 m.

Let us now consider the vertical shear of the transverse velocity, $v$, which is induced by the earth’s rotation, see Figure 4. The fact that the signal has not yet become periodic is, as we noticed above, largely due to the earth’s rotation: and the lack of periodicity is therefore more clearly visible in Figure 4 than in Figure 3. Remarkably, $\bar{v}$ follows the overall profile of the internal tide; and is scarcely affected by the presence of the solitary waves. This can be understood from the expression for the potential vorticity, $\eta \sim \bar{v}_x$ (at lowest order, see Gill, 1982) which indicates that a peak in $\eta$ is accompanied by a change in the derivative of $\bar{v}$, rather than by a change in $\bar{v}$ itself. Indeed, these wiggles in $\bar{v}$, though weak, are visible in Figure 4.

Next, we consider the generation of the internal tide near the shelf break, see Figure 5. During the first half of the ebb phase, a depression is formed that travels seaward, and gradually disintegrates into solitary waves. During the second half of the ebb phase, another depression is formed, which travels on-shelf and also disintegrates. The latter, of course, is the depression that we recorded at $x = 6.0$ in Figure 3. Interestingly, Pingree et al. (1986) hypothesized that a depression is formed during the ebb phase, which splits up into two depressions, one traveling seaward, the other on-shelf. However, we do not observe the split-up of one single depression; instead we observe the generation of two separate depressions. Notice that the solitary waves traveling seaward have a longer length scale than those traveling on-shelf. This is because the depth differs considerably (being 4000 m and 170 m, respectively). From the expression for the length scale of the KdV-soliton in a two-layer system

$$\frac{a}{q^2} = \frac{4(h_1h_2)^2}{3(h_1 - h_2)}$$
Figure 5. The generation of the internal tide in the Celtic Sea during spring tides, between 4½ and 5½ tidal periods; the upper part of the figure shows the ebb phase, the lower half the flood phase. The lapse of time between successive figures is ½a tidal period. The vertical offset in $\eta$ is -2.
where \( a \) is the amplitude and \( 1/q \) the width, we observe that the square of the width is proportional to the thickness of the lower layer (assuming for simplicity that the lower layer is much thicker than the upper one). This implies that the ratio of the widths should be about 5:1 in the present case, which is in agreement with Figure 5. Finally, we notice that the barotropic tidal advection plays no role (at least not visibly) so far as the seaward traveling waves are concerned. By contrast, those traveling on-shelf are clearly advected; in particular, during maximum ebb (between \(-12\) and \(-15\), on the vertical) they seem not to move at all. This implies that their speed is about 0.7 m/s (since this is the maximum current speed of the barotropic tide), which is in excellent agreement with estimates by Pingree and Mardell.

Let us now consider the generation of internal tides during neap tides \((F = 0.27, \epsilon = 0.065)\), see Figures 6 and 7. Now there are clearly no solitary waves; the profile looks like a slightly deformed sinusoidal wave. For the amplitude (trough to crest) we find about 13 m, whereas the observed values lie between 13 and 17 m (Fig. 1, 16–18 July); apart from small oscillations of short periods, their profile looks fairly sinusoidal, too.

We repeat the neap-tide experiment, but this time without the earth’s rotation \((\mu = 0)\), see Figure 8. Clearly, the internal tide now disintegrates into solitary waves. We can thus conclude that in the experiment of Figures 6 and 7, which corresponds to neap-tide conditions in the Celtic Sea, it is the earth’s rotation (Coriolis dispersion) that prevents the internal tide from disintegrating.

A clue to the interpretation of these results can be found by considering the nonlinear Klein-Gordon equation, which follows from (32) to (34) by setting \( \delta = 0 \) (hydrostatic approximation), \( U, V, W = 0 \) (no barotropic tidal flow), \( h = 1 \) (shelf regime):

\[
\eta_t - \gamma^2 \eta_{xx} + 3\tilde{\epsilon}(\eta\eta_x)_x + \mu \eta = 0
\]

where \( \tilde{\epsilon} = \epsilon(1 - 2\alpha - \epsilon_b)/(1 - \epsilon_b) \) and \( \gamma^2 = \alpha(1 - \alpha - \epsilon_b)/(1 - \epsilon_b)^2 \). It can be shown (Ostrovsky, 1978; also, more detailed, Gerkema, 1994, §2.4) that this equation admits form-preserving periodic solutions only if rotation is sufficiently strong in comparison with nonlinearity, i.e. (in a first approximation) if \( \tilde{\epsilon}/\mu < \gamma^2/6a_c \), where \( a_c \) is the height of the crest.

It is tempting to apply this result to the present, more complicated problem by assuming that no disintegration takes place if the parameters satisfy this inequality. Using the inequality (with \( a_c \approx 0.5 \), see Figs. 3 and 7), we find that, in order that the earth’s rotation may prevent the disintegration of the internal tide, \( \mu \) should have the value 0.9 at spring tides, and 0.4 at neap tides. The actual value of \( \mu \) is 0.59. This implies that the earth’s rotation is too weak at spring tides (hence disintegration), but strong enough at neap tides (no disintegration), which is precisely what we found in the numerical experiments.

To summarize, during neap tides the internal tide is characterized by a balance
Figure 6. The generation of the internal tide in the Celtic Sea during neap tides, between 4½ and 5½ tidal periods; the upper part of the figure shows the ebb phase, the lower half the flood phase. The lapse of time between successive figures is ¼ tidal period. The vertical offset in $\eta$ is −2.
between nonlinearity and Coriolis dispersion on the long internal-tide lengthscale. By contrast, during spring tides no such balance can exist because nonlinearity is too strong; instead a balance between nonlinearity and nonhydrostatic dispersion is established on the short solitary-wave lengthscale.

b. Massachusetts Bay. In the past decades several observations on internal tides and internal solitary waves have been made in Massachusetts Bay, e.g. Halpern (1971), Haury et al. (1979), and Chereskin (1983). The internal tides are generated by a barotropic tidal flow over Stellwagen Bank, and the solitary waves are due to a fission of the internal tides. The present study can be regarded as being complementary to an earlier study by Hibiya (1988), who dealt with the generation of internal waves over the bank, showing that this generation process cannot be regarded as quasi

Figure 7. The elevation of the interface at position B (25 km from the break), during neap tides. One unit in $\eta$ corresponds to about 11 m.

Figure 8. The elevation of the interface at position B (25 km from the break), during neap tides, without the earth's rotation. One unit in $\eta$ corresponds to about 11 m.
We focus on the later development of the internal tide during the course of its propagation.

We focus on Halpern (1971), see Figure 9. He measured at a location about 9 km to the west of Stellwagen Bank ('site T', 70°24.5'W 42°16.5'N), where the depth is 82 m. The height of the bank is 55 m, so $c = 55/82 = 0.67$. The analysis in Halpern (1971) provides no unequivocal clue to obtaining the parameters of the two-layer system, but it seems reasonable to take 16 m as the thickness of the upper layer (thus $\alpha = 16/82 = 0.195$), and $\Delta \rho = 0.001$; hence, from Eq. (3), $c = 0.52$ m/s. The frequency of the M2-tide is $\sigma = 1.41 \cdot 10^{-4}$ rad/s. Hence one unit in $x$ corresponds to

$h(\Lambda x) \uparrow$

site $T$

Stellwagen Bank

Figure 10. The topography of Stellwagen Bank, and 'site $T$', where we record the evolution of the internal tide.
3.7 km (= c/σ). Furthermore, we obtain for the parameters of nonhydrostatic and Coriolis dispersion: δ = 0.00049 and μ = 0.48, respectively.

It is not clear where exactly the waves observed at 'site T' are generated. Halpern gives a bathymetric cross section along the latitude at which site T is located; here the bank is not very steep and our simulations based on this topography were not in good agreement with the observations. Taking instead the transect along which Haury et al. (1979) made their records (in a northeastern direction from site T) seems to give better results. We use the following topography:

$$h(X) = \begin{cases} 
0 & X \leq -\frac{4\pi}{9} \\
\frac{1}{2}(1 + \cos (9X/4)) & -\frac{4\pi}{9} \leq X \leq 0 \\
\frac{1}{2}(1 + \cos (27X/40)) & 0 \leq X \leq \frac{40\pi}{27} \\
0 & X \geq \frac{40\pi}{27}
\end{cases}$$

where X = Λx. With these choices, h, h_x = O(1), and the east-side of the bank is about thrice as long as the west-side (see Fig. 10). The estimated length of the west-side slope is 3.5 km, which corresponds to 0.95 units in x, and to 4π/9 in terms of X. Hence Λ = X/x = (4π/9)/0.95 = 1.5.

The tidal flow in Massachusetts Bay is strongly asymmetric; the ebb (i.e. eastward) current is about twice as strong as the flood current (Chereskin, 1983). We therefore simulate the flow by re-defining U as follows

$$U = \frac{1}{5} \frac{1 - \varepsilon_b}{1 - \varepsilon_b h} (\cos 2(t + \phi) - 4 \cos (t + \phi))$$

and V and W accordingly, with φ = 1.79748 (the phase φ ensures that U = 0 at t = 0). The maximum cross flow over the top of the bank is about 0.4 m/s, so $F = 0.77$. Thus we find for the parameter of nonlinearity: $ε = 0.77 (= F\Lambdaε_b)$. We use the following space and time steps: $Δx = 0.002$ and $Δt = 0.0005$, corresponding to dimensional values of about 7.4 m and 3.5 s, respectively.

The actual simulation of Halpern’s observation is shown in Figure 11 (the vertical axis being reversed, for convenience). We only show one tidal period; during subsequent periods the signal is almost the same. The shape of the profile is qualitatively similar to Halpern’s (see Fig. 9). Moreover, we find for the amplitude of the first solitary wave a value between 16 m and 19 m (depending on whether the height of the front or of the back is measured), whereas the real observation yields values of about 15 m. The first five solitary waves have an average period of 6 min, which is in good agreement with the real observation, where the estimated value lies between 6 and 8 min. There are about 20 solitary waves; in reality there are perhaps more (see Fig. 9), although it is difficult to judge how many of the smaller ones should be counted as solitary wave, and how many as noise (Chereskin regards 30 as the
upper estimate). For the distance between successive solitary waves we find about 200 m; Halpern’s estimate lies between 180 and 230 m (based on observations of the ripples at the upper surface, which accompany the large solitary waves).

It is also interesting to consider $\bar{v}$ (the vertical shear of the horizontal velocity component in the transverse direction), see Figure 12 (also with reversed vertical axis). This component would be absent if rotation were absent. The component follows the overall profile of the internal tide; the solitary waves themselves are only visible as small wiggles. As explained above, this can be understood from the expression for the potential vorticity, $\eta \sim \bar{v}_x$, which shows that a large elevation of $\eta$ is accompanied by a change in the derivative of $\bar{v}$.

In Figure 13 is shown the profile while passing the location $x = -4$, that is, about 15 km from the bank. The number of solitary waves is about the same as in Figure 11,
so we can regard the profile at site T as fully developed. The main difference consists in the spreading of the sequence; the larger solitary waves travel faster, and thus leave the smaller ones behind them; this is a manifestation of nonlinearity.

Finally, we consider the influence of the earth's rotation. We repeat the experiment shown in Figure 11, but now with $\mu = 0$, *no earth's rotation*, see Figure 14. The difference from Figure 11 is striking; there are now much more solitary waves. So we arrive at the same conclusion as in the Celtic Sea experiment, namely that the earth's rotation is an important factor in the evolution of the internal tide, in that it tends to impede the disintegration into solitary waves.

Some analytical results can be obtained in the case $\mu = 0$ (no rotation). By using the inverse scattering technique for the KdV equation, one can derive an expression (in terms of the forcing parameters) for the number of solitary waves that evolve.

![Figure 13](image13.png)

*Figure 13. A sequence of solitary waves passing a location 15 km from Stellwagen Bank (vertical axis reversed). One unit in $\eta$ corresponds to 21 m.*

![Figure 14](image14.png)

*Figure 14. A sequence of solitary waves passing site T, now *without the earth's rotation* (vertical axis reversed). One unit in $\eta$ corresponds to 21 m.*
from the internal tide (Gerkema, 1994; Gerkema and Zimmerman, 1995). That number is given by the largest integer satisfying:

$$N < \frac{1}{2} \left[ \left( 1 + \frac{6\alpha(1-2\alpha)}{\delta\alpha(1-\alpha)} \right)^{1/2} + 1 \right]$$  \hspace{1cm} (37)

where $a$ is the crest-trough distance of the internal tide at its initial stage. This formula shows, in accordance with one's expectations, that more solitary waves appear in case of stronger forcing (i.e., larger $\epsilon$), or shallower seas (i.e., smaller $\delta$).

Here $a \approx 1.1$, whence we find $N \approx 60$. Although this value is not yet reached at site T (Fig. 14), farther away from the bank we record between 50 and 60 solitary waves in each tidal period, see Figure 15. Interestingly, this number accords well with the estimate of 60 solitary waves by Maxworthy (1979), who used an empirically obtained profile (over Stellwagen Bank) as the initial profile for solving the KdV equation. He incorrectly considered his outcome to agree with the empirical records; it is now clear that it was due to his neglect of the earth's rotation why he found a number that was far too large.

By comparing either Figures 11 and 14, or Figures 13 and 15, we observe that the earth's rotation increases the phase speed of the solitary waves, for they arrive earlier at the location in question when rotation is present. More precisely, we find that the largest solitary wave, traveling from site T to the station 15 km from Stellwagen Bank, has an average speed of 0.53 m/s if rotation is present, and 0.45 m/s if rotation is absent.4

In the latter case we may expect the KdV theory to be valid. The expression for the

---

4. These values represent the proper phase speeds of the waves; we have corrected for the barotropic tidal advection.
phase speed of a KdV-soliton in a two-layer system reads

\[ C = c_0 \sqrt{1 + \frac{h_1 - h_2}{2h_1h_2} (a_1 + 2a_2)} \]  

(38)

where \( c_0 \) is given by (35); \( a_1 \) denotes the vertical position of the trough of the solitary wave, \( a_2 - a_1 \) its vertical extension. Using this formula, we find that a solitary wave having the observed amplitude should travel at a speed of 0.47 m/s, which is not too far off the value obtained from Figures 14 and 15 (0.45 m/s). This value is well above the phase speed of a linear internal tide in a nonrotating system (0.36 m/s), but lower than the linear phase speed in a rotating system (0.50 m/s). Contrary to one's expectations, the same formula even works in the case with rotation, if \( a_1 \) and \( a_2 \) are chosen in accordance with Figures 11 and 13. We then find 0.54 m/s, which is very close to the value obtained from Figures 11 and 13 (0.53 m/s).

4. Summary and discussion

In this paper a new set of forced rotation-modified Boussinesq equations was derived which describes the generation of nonlinear dispersive internal tides by a barotropic tidal flow over a large-amplitude topographic feature and their later evolution during the course of propagation. Coriolis effects are included by using the f-plane approximation. Restrictions on the range of applications are due to the representation of the density stratification by a two-layer model, and the assumption of no spatial variation in the transverse direction.

The simulations of the observations in the Celtic Sea and Massachusetts Bay were quite satisfactory, and we may conclude that the model (32) to (34) provides a suitable tool for the description of the generation and fission of nonlinear dispersive internal tides. The amplitudes were perhaps a bit too large, which may be ascribed to the neglect of friction in the model.

We found that the earth's rotation is a most important factor in the evolution of the internal tide in that it tends to impede the fission of the internal tide into solitary waves. In the Celtic Sea experiment at neap tides, rotation was sufficiently strong to prevent the fission, while in Massachusetts Bay it was not strong enough to do so, but nevertheless reduced the number of emerging solitary waves considerably.

The underlying idea is that Coriolis dispersion can check the steepening effect of nonlinearity if \( \mu/\epsilon \) is sufficiently large. In that case the internal tide is governed by a balance between nonlinearity and Coriolis dispersion on the long internal-tide lengthscale, and no disintegration occurs. If Coriolis dispersion is not strong enough, then nonlinearity makes the internal tide steepen until nonhydrostatic dispersion becomes important, and a fission into solitary waves occurs. When the process of disintegration has come to an end, a balance between nonlinearity and nonhydrostatic dispersion exists on the short solitary-wave lengthscale.

Leonov's result (1981) that solitary waves cannot exist in the presence of rotation...
seems contradictory to our numerical solutions, in which such waves seem to appear. Upon closer consideration, however, there is no contradiction, because Leonov's analysis was based on the premise that the strength of nonlinearity, and nonhydrostatic and Coriolis dispersion is of equal order of magnitude. This assumption fails to apply to the oceanic internal solitary waves because on their scale the effect of the earth's rotation is very weak (see below), while nonlinearity and nonhydrostatic dispersion dominate. Hence, the direct influence of the earth's rotation on the solitary waves is so small as to be negligible.

The real, oceanic situation is rather more complicated, though, because there is one more length scale involved: the long internal-tide scale, on which the earth's rotation plays an important role. In an indirect and complicated way, it may also affect the solitary waves (if they appear), because they are part of the larger structure; but this problem is beyond the scope of Leonov's analysis.

A hint of this effect can be traced from the analysis of the phase speeds in the previous section. The (dimensional) dispersion relation for linear Poincaré waves reads $\sigma^2 = f^2 + c_0^2 k^2$. The solitary waves in Massachusetts Bay have a length scale of 200 m; hence, the first term on the right-hand side is a factor $10^{-4}$ smaller than the second term, which implies that on a linear level the phase speed ($\sigma/k$) is not significantly modified by rotation. However, due to the fact that the solitary waves are part of a large-scale structure at which rotation is significant, their amplitudes $a_1$ and $a_2$ are found to depend on $\mu$ (this, at least, is suggested by the numerical experiments), and since on a nonlinear level the phase speed depends on the amplitude (see (38)), the earth's rotation does affect the phase speed of the solitary waves, though in an indirect way. We think that these ideas can be substantiated mathematically, but this problem is beyond the scope of the present study.

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APPENDIX

Derivation of model equations

We choose the (small) parameters in accordance with (2) and (5). Notice that

$$h_x = \Lambda h_X = O(\Lambda) = O(\epsilon^{1/2})$$

$$h_{xx} = \Lambda^2 h_{XX} = O(\Lambda^2) = O(\epsilon)$$

and so on, and similarly for $U$, $V$ and $W$, since their $x$-dependence is wholly due to variations in $h$. This is a point to be kept in mind during the derivation that follows: by differentiating $h$, $U$, $V$ or $W$ to $x$, their order is increased by one (i.e. $\epsilon^{1/2}$).
We develop the variables $u_i, v_i, w_i, p_i$ and $\eta$ in a perturbation series, in which $\epsilon^{1/2}$ acts as the small parameter. Thus

$$\zeta = \zeta^{(0)} + \epsilon^{1/2}\zeta^{(1)} + \epsilon\zeta^{(2)} + \cdots$$

where $\zeta$ stands for any of the variables. We also write

$$W^* \equiv W|_{z=a(1-\epsilon_b)\eta} = W^{(0)} + \epsilon W^{(2)}$$

with

$$W^{(0)} = \frac{\alpha \sin t}{(1 - \epsilon_b h)^2} h_x$$

$$W^{(2)} = -\frac{(1 - \epsilon_b)\eta \sin t}{(1 - \epsilon_b h^2)} h_x.$$

In fact, $W^{(0)} = W_{z=0}$, which represents the forcing in the linear problem, where the amplitude of the internal tide, i.e. the displacement of the interface, is infinitesimal. As soon as we consider nonlinear internal tides, we should allow for the fact that the amplitude is finite; this is done by including $W^{(2)}$.

For convenience, we introduce

$$F_* = \frac{F}{\epsilon^{1/2}}, \quad \Lambda_* = \frac{\Lambda}{\epsilon^{1/2}}, \quad \mu_* = \frac{\mu}{\epsilon^{1/2}}, \quad \delta_* = \frac{\delta}{\epsilon}$$

which are all of order one.

We successively derive the equations of order $\epsilon^0$, order $\epsilon^{1/2}$ and order $\epsilon$. Finally, the three orders are combined to yield the forced rotation-modified Boussinesq equations. It is worth while to describe briefly what the equations at subsequent orders stand for:

**Zeroth order ($\epsilon^0$):** The generation of linear, hydrostatic internal tides. No rotation, no barotropic tidal advection.

**First order ($\epsilon^{1/2}$):** The generation of linear, hydrostatic internal tides in a rotating system. Barotropic tidal advection is included (the equations may therefore be termed ‘quasi-nonlinear’).

**Second order ($\epsilon$):** The generation of nonlinear, nonhydrostatic internal tides in a rotating system. Barotropic tidal advection is included.

In all cases, the bottom topography is of finite amplitude.

At all orders considered here, the vertical shear of the horizontal velocities, defined in (11) and (12), takes the following form:

$$\langle \overline{u^{(n)}}, \overline{v^{(n)}} \rangle = \lim_{z \uparrow 0} (u^{(n)}_2, v^{(n)}_2) - \lim_{z \downarrow 0} (u^{(n)}_1, v^{(n)}_1).$$  

The following derivation is given in more detail in (Gerkena, 1994).
Zeroth order

At zeroth order (i.e. \( \epsilon^0 \)), (21) to (24) read:

\[
\begin{align*}
\quad u_{i,t}^{(0)} + p_{i,x}^{(0)} &= 0 \quad \text{(42)} \\
\quad v_{i,t}^{(0)} + u_i^{(0)} &= 0 \quad \text{(43)} \\
\quad p_i^{(0)} &= 0 \quad \text{(44)} \\
\quad u_{i,x}^{(0)} + (1 - \epsilon_b) w_{i,z}^{(0)} &= 0 \quad \text{(45)}
\end{align*}
\]

and the boundary conditions, (25) to (28):

\[
\begin{align*}
\eta^{(0)} &= 0 \quad \text{at } z = \alpha \quad \text{(46)} \\
\eta^{(0)} &= 0 \quad \text{at } z = \alpha - 1 + \epsilon_b h \quad \text{(47)} \\
W^{(0)} + w_i^{(0)} &= \eta_i^{(0)} \quad \text{at } z = 0 \quad \text{(48)} \\
\eta^{(0)} &= p_i^{(0)} - p_i^{(0)} \quad \text{at } z = 0. \quad \text{(49)}
\end{align*}
\]

It follows that \( p_1, p_2, u_1, u_2, v_1 \) and \( v_2 \) are independent of \( z \) (quasi-hydrostatic state).

From (42), (43), (49) and (41) follows:

\[
\begin{align*}
\ddot{u}_t^{(0)} + \eta_t^{(0)} &= 0 \quad \text{(50)} \\
\ddot{v}_t^{(0)} + \ddot{u}^{(0)} &= 0. \quad \text{(51)}
\end{align*}
\]

From (45) to (48) one easily obtains

\[
\eta_t^{(0)} - W^{(0)} + \frac{\alpha(1 - \alpha - \epsilon_b h)}{(1 - \epsilon_b)(1 - \epsilon_b h)} \ddot{u}_t^{(0)} = 0. \quad \text{(52)}
\]

Eqs. (50) and (52), the zeroth-order evolution equations, together describe linear, hydrostatic internal tides in a non-rotating system, generated by an infinitesimally weak barotropic flow over a finite-amplitude bottom topography. At this order, they are uncoupled from the second momentum equation, (51).

One can also derive the following expressions, which are useful at higher orders:

\[
\begin{align*}
\dot{u}_1^{(0)} &= \frac{\alpha - 1 + \epsilon_b h}{1 - \epsilon_b h} \ddot{u}^{(0)} \quad \text{(53)} \\
\dot{u}_2^{(0)} &= \frac{\alpha}{1 - \epsilon_b h} \ddot{u}^{(0)} \quad \text{(54)} \\
\dot{w}_1^{(0)} &= \left(1 - \frac{z}{\alpha}\right) \left[ \eta_t^{(0)} - W^{(0)} \right] \quad \text{(55)}
\end{align*}
\]
First order

At first order (i.e. $\epsilon^{1/2}$), (21) to (24) read:

\[ u_{ix}^{(1)} + F_\ast U u_{i,x}^{(0)} - \mu_* v_i^{(0)} + p_{i,x}^{(1)} = 0 \]  
(59)

\[ v_{ix}^{(1)} + F_\ast U v_{i,x}^{(0)} + u_i^{(1)} = 0 \]  
(60)

\[ p_{i,x}^{(1)} = 0 \]  
(61)

\[ u_{i,x}^{(1)} + (1 - \epsilon_b)w_{i,x}^{(1)} = 0 \]  
(62)

and the boundary conditions, (25) to (28):

\[ w_1^{(1)} = 0 \quad \text{at} \quad z = \alpha \]  
(63)

\[ (1 - \epsilon_b)w_2^{(1)} = \Lambda_\ast \epsilon_b \mu u_2^{(0)} h_x \quad \text{at} \quad z = \alpha - 1 + \epsilon_b h \]  
(64)

\[ w_i^{(1)} = \eta_i^{(1)} + F_\ast U \eta_x^{(0)} \quad \text{at} \quad z = 0 \]  
(65)

\[ \eta^{(1)} = p_2^{(1)} - p_1^{(1)} \quad \text{at} \quad z = 0. \]  
(66)

It follows that $p_1, p_2, u_1, u_2, v_1$ and $v_2$ are still independent of $z$. From (59), (60), (66) and (41) follows:

\[ \tilde{u}_t^{(1)} + F_\ast U \tilde{u}_x^{(0)} - \mu_* \tilde{v}^{(0)} + \eta_x^{(1)} = 0 \]  
(67)

\[ \tilde{v}_t^{(1)} + F_\ast U \tilde{v}_x^{(0)} + \tilde{u}_x^{(1)} = 0. \]  
(68)

Eqs. (54), and (62) to (65) imply:

\[ \eta_t^{(1)} + F_\ast U \eta_x^{(0)} + \frac{\alpha(1 - \alpha - \epsilon_b h)}{(1 - \epsilon_b)(1 - \epsilon_b h)} \tilde{u}_x^{(1)} - \frac{\Lambda_\ast \alpha^2 \epsilon_b \mu u_2^{(0)} h_x}{(1 - \epsilon_b)(1 - \epsilon_b h)^2} = 0 \]  
(69)

Clearly, the advection by the barotropic tide is included at this order.

In view of later usage we note that

\[ u_2^{(1)} = \frac{\alpha}{1 - \epsilon_b h} \tilde{u}^{(1)}. \]  
(70)
Second order

At second order (i.e. \( \epsilon \)), (21) to (24) read:

\[
\begin{align*}
\frac{\partial u^{(2)}_{x}}{\partial z} + F_* U u^{(1)}_{x} + F_* \Lambda_{*} u^{(0)}_{x} U_x + u^{(0)}_{x} u^{(0)}_{x} - \mu_* v^{(1)}_{x} + p_{i,x}^{(2)} &= 0 \quad (71) \\
\frac{\partial v^{(2)}_{x}}{\partial z} + F_* U v^{(1)}_{x} + F_* \Lambda_{*} u^{(0)}_{x} V_x + u^{(0)}_{x} v^{(0)}_{x} + u^{(2)}_{x} &= 0 \quad (72) \\
\delta_{*} (1 - \epsilon_{b}) w^{(0)}_{i,x} &= -p_{i,x}^{(2)} \quad (73) \\
u^{(2)}_{i,x} + (1 - \epsilon_{b}) w^{(2)}_{i,z} &= 0 \quad (74)
\end{align*}
\]

and the boundary conditions, (25) to (28):

\[
\begin{align*}
w^{(2)}_{1} &= 0 \quad \text{at } z = \alpha \quad (75) \\
(1 - \epsilon_{b}) w^{(2)}_{2} &= \Lambda_{*} \epsilon_{b} u^{(1)}_{x} h_x \quad \text{at } z = \alpha - 1 + \epsilon_{b} h \quad (76) \\
\mathcal{W}^{(2)} + w^{(2)}_{1} &= \eta^{(2)}_{1} + F_* U \eta^{(1)}_{x} + (u^{(0)}_{x} \eta^{(0)})_{x} \quad \text{at } z = 0 \quad (77) \\
\eta^{(2)}_{1} &= p^{(2)}_{2} - p^{(2)}_{1} \quad \text{at } z = 0. \quad (78)
\end{align*}
\]

At this order \( p_1, p_2, u_1, u_2, v_1 \) and \( v_2 \) are no longer independent of \( z \), which implies that nonhydrostatic effects are now being included.

From (71), (72), (78) and (41) follows:

\[
\begin{align*}
\frac{\partial \bar{u}^{(2)}_{i}}{\partial z} + F_* U \bar{u}^{(1)}_{x} + F_* \Lambda_{*} \bar{u}^{(0)} U_x + \frac{2\alpha - 1 + \epsilon_{b} h}{1 - \epsilon_{b} h} \bar{u}^{(0)} \bar{u}^{(0)} - \mu_* \bar{v}^{(1)}_{x} + \eta^{(2)}_{x} &= 0 \quad (79) \\
\frac{\partial \bar{v}^{(2)}_{i}}{\partial z} + F_* U \bar{v}^{(1)}_{x} + F_* \Lambda_{*} \bar{u}^{(0)} V_x + \frac{2\alpha - 1 + \epsilon_{b} h}{1 - \epsilon_{b} h} \bar{u}^{(0)} \bar{v}^{(0)} + \bar{u}^{(2)} &= 0 \quad (80)
\end{align*}
\]

where we used the zeroth-order expressions for \( u^{(0)}_{i} \) and \( v^{(0)}_{i} \), in order to eliminate them from the nonlinear terms in favor of \( \bar{u}^{(0)} \) and \( \bar{v}^{(0)} \).

The third equation is less easily obtained at this order, so we give more intermediate steps. We differentiate (71) to \( z \),\(^5\) and (73) to \( x \), and combine the ensuing expressions:

\[
u^{(2)}_{i,x} = \delta_{*} (1 - \epsilon_{b}) w^{(0)}_{i,x}. \quad (81)
\]

This expression and (74) together imply

\[
w^{(2)}_{i,zz} = -\delta_{*} w^{(0)}_{i,\alpha}. \quad (82)
\]

5. Recall that \( u^{(0)}_{i}, u^{(1)}_{i} \) and \( v^{(1)}_{i} \) are independent of \( z \), and thus all vanish.
In the right-hand side we insert (55) for \( i = 1 \), and (56) for \( i = 2 \):

\[
\begin{align*}
    \omega_{1,zz}^{(2)} &= -\delta \left( 1 - \frac{z}{\alpha} \right) \eta_{\alpha i}^{(0)} \\
    \omega_{2,zz}^{(2)} &= -\delta \left( 1 + \frac{z}{1 - \alpha - \epsilon_b h} \right) \eta_{\alpha i}^{(0)}.
\end{align*}
\]

The \( \mathcal{W}^{(0)} \) term vanished, because, by differentiating twice to \( x \), its order is increased by two—see (39). After integrating twice with respect to \( z \), and using (77), we obtain

\[
\begin{align*}
    \omega_{1}^{(2)} &= \frac{1}{2} \delta \left( z^2 - \frac{z^3}{3\alpha} \right) \eta_{\alpha i}^{(0)} - \frac{z}{1 - \epsilon_b} \left( u_{1,zz}^{(2)}|_{z=0} \right) + \eta_i^{(2)} + F_{\ast} U \eta_i^{(1)} \\
    &\quad + \left( u_{1}^{(0)} \eta_{\alpha i}^{(0)} \right)_x - \mathcal{W}^{(2)} \\
    \omega_{2}^{(2)} &= -\frac{1}{2} \delta \left( z^2 + \frac{z^3}{3(1 - \alpha - \epsilon_b h)} \right) \eta_{\alpha i}^{(0)} - \frac{z}{1 - \epsilon_b} \left( u_{2,zz}^{(2)}|_{z=0} \right) + \eta_i^{(2)} + F_{\ast} U \eta_i^{(1)} \\
    &\quad + \left( u_{2}^{(0)} \eta_{\alpha i}^{(0)} \right)_x - \mathcal{W}^{(2)}.
\end{align*}
\]

We take the limit \( z \to \alpha \) in (83), and \( z \to \alpha - 1 + \epsilon_b h \) in (84). Using (75) and (76), and combining the ensuing expressions, we find,

\[
\begin{align*}
    \eta_i^{(2)} + F_{\ast} U \eta_i^{(1)} - \mathcal{W}^{(2)} + \frac{2\alpha - 1 + \epsilon_b h}{1 - \epsilon_b h} \left( \tilde{u}(0) \eta_{\alpha i}^{(0)} \right)_x + \frac{\alpha(1 - \alpha - \epsilon_b h)}{(1 - \epsilon_b)(1 - \epsilon_b h)} \tilde{u}_x \\
    &\quad - \frac{\Lambda_{\ast} \alpha^2 \epsilon_b \tilde{u}_x}{(1 - \epsilon_b)(1 - \epsilon_b h)^2} - \frac{1}{3} \delta \alpha(1 - \alpha - \epsilon_b h) \eta_{\alpha i}^{(0)} = 0
\end{align*}
\]

where we substituted (54) and (53) in the nonlinear terms, in order to eliminate \( u_{1}^{(0)} \) in favor of \( \tilde{u}^{(0)} \); we also used (70) to express the penultimate term in (85) in terms of \( \tilde{u}^{(1)} \).

**Combining the lowest three orders**

In this section we combine the results of the zeroth, first and second order, i.e. we calculate (50) + \( \epsilon^{1/2} \times (67) + \epsilon \times (79) \), (51) + \( \epsilon^{1/2} \times (68) + \epsilon \times (80) \) and (52) + \( \epsilon^{1/2} \times (69) + \epsilon \times (85) \). This yields

\[
\begin{align*}
    \tilde{u}_t + F \tilde{U} \tilde{u}_x + F \tilde{U} \tilde{u}_x + \epsilon \frac{2\alpha - 1 + \epsilon_b h}{1 - \epsilon_b h} \tilde{u}_x = O(\epsilon^{3/2}) \\
    \tilde{v}_t + F \tilde{U} \tilde{v}_x + F \tilde{U} \tilde{v}_x + \epsilon \frac{2\alpha - 1 + \epsilon_b h}{1 - \epsilon_b h} \tilde{v}_x = O(\epsilon^{3/2}) \\
    \eta_t + F \eta_x - \mathcal{W}^{(e)} + \epsilon \frac{2\alpha - 1 + \epsilon_b h}{1 - \epsilon_b h} \left( \tilde{u} \eta \right)_x + \frac{\alpha(1 - \alpha - \epsilon_b h)}{(1 - \epsilon_b)(1 - \epsilon_b h)} \tilde{u}_x \\
    &\quad - \frac{\alpha^2 \epsilon_b \tilde{u}_x}{(1 - \epsilon_b)(1 - \epsilon_b h)^2} - \frac{1}{3} \delta \alpha(1 - \alpha - \epsilon_b h) \eta_{\alpha i} = O(\epsilon^{3/2})
\end{align*}
\]
where we used the identity $h_t = \Delta h_x$, and (40).

These equations, however, suffer from a lack of symmetry. This is most readily seen by considering the case of an even bottom (so that $h_t, U_x, V_x, W = 0$). Then we should be able to make a transformation to the frame of reference in which we move along with the barotropic tide, i.e., we replace $\cdot_t$ with $\cdot - FU(\cdot)$ (the dot stands for $\bar{u}, \bar{v}$ or $\eta$). In the new frame of reference there is of course no barotropic flow, thus no terms containing $U$ should be left. This is indeed the case for (32) and (33), but not for (88), in which the term $\frac{1}{2}\delta a(1 - \alpha - \epsilon_b h)F(U_{\eta_x})_{xx}$ would appear. This term destroys the symmetry; clearly, we can restore the symmetry by including

$$-\frac{1}{2}\delta a(1 - \alpha - \epsilon_b h)F(U_{\eta_x})_{xx}$$

on the left-hand side of (88). This is in fact a term that would have appeared at the next order. In preliminary numerical experiments performed without (89), the solutions were often spoiled by illicit dispersive effects. By including (89), we obtained physically acceptable solutions. The improved version of (88) thus reads:

$$\eta_t + FU\eta_x - W^s + \epsilon \frac{2\alpha - 1 + \epsilon_b h}{1 - \epsilon_b h} (\bar{u}\eta)_x + \frac{\alpha(1 - \alpha - \epsilon_b h)}{(1 - \epsilon_b h)(1 - \epsilon_b h)} \bar{u}_x$$

$$- \frac{\alpha^2 \epsilon_b \bar{u}_h h_x}{(1 - \epsilon_b)(1 - \epsilon_b h)^2} - \frac{1}{3} \delta a(1 - \alpha - \epsilon_b h)[\eta_t + FU\eta_x]_{xx} = O(\epsilon^{3/2}).$$

Finally, upon neglecting the right-hand sides of (86), (87) and (90), we obtain (32) to (34).

REFERENCES


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